Gold accumulation and wealth dynamics across generations

Project report submitted to IGPC, IIMA.

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March 23, 2018

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1

Gold as an asset: Composition of households asset portfolio

Asset portfolio construction is one of the most fundamental economic activities that households execute.[1] Let us motivate the basic reason with a simple economic argument. The standard wisdom is that people do not like fluctuations in consumption. Note that it does not mean that they dislike fluctuations in income! A basic formulation of a life-cycle savings problem is as follows. A household lives for two periods, present (P) and future (F). Let us say the household utility function is defined over consumption (C) as

$$U(C_P, C_F) = u(C_P) + \beta U(C_F), \tag{1.1}$$

where β is the subjective discount factor. The two-period budget constraint is

$$C_P + \frac{C_F}{1+r} = y_1 + \frac{y_2}{1+r}, \tag{1.2}$$

where r is the risk-free interest rate and $\{y_1, y_2\}$ is the income stream. A simple optimization shows that (under reasonable assumptions on curvature of $u(.)$), this household will try to save in the present period, if the current income exceeds the consumption demand and the future income is not deemed sufficient.

This simple model makes an innocuous assumption that the household can access a financial market (for both borrowing and lending) if need be. In reality, that appears to be a problematic issue. Most of the less developed economies are characterized by chronic lack of financial markets. Usually, countries like India have vast population who are out of reach of the for-

[1] This chapter is based on some primary literature survey that Swarna Parameswaran (FPM program, IIMA) had conducted under my supervision as a summer project (2017).

mal financial market. Thus the consumption smoothing mechanism works differently for households in rich countries vis-a-vis poor countries.

In such cases, households will resort to alternatives. Village economies are characterized by informal risk-sharing through insurance among households. However, they also rely on tangible assets like land or gold for multiple economic reasons. Often, these are used as collaterals for getting loans even within the informal market. Gold has a special role in this regard. Other physical collaterals like land, have intrinsic values for productive purpose. Gold is an asset which has no productive usage (at least for rural or urban households).

Gold as an asset is also special because of its social influence. For last hundred years, the price of gold has not declined substantially and it has emerged as a safe asset, where risk is apparently lower bounded. Due to interaction of the economic argument (gold as a store of value) and social developments (accumulation of gold confers higher prestige), gold has also become a measure of the well known phenomenon of *keeping up with the Joneses*.

1.1 Literature

The literature on economic and social effects of gold is not very extensive. Let us first discuss the wealth inequality scenario and then we will discuss the role gold in the economic and social context. Subramanian and Jayaraj (2006) (see Ref. [8]) did a comprehensive analysis of the evolution of wealth inequality (1961-62 to 2002-03) within India. They used the data from All India Debt and Investment Surveys constructed by the National Statical Survey organization. They made a general observation was that the real wealth increased for the average households. Both urban and rural households grew the real wealth. Simultaneously, debt-holding also showed an improvement. Interestingly, debt-holding turned out to be inversely related to the size of asset holdings. This possibly indicates a feature of the households that they are treating debt and assets as substitutes for providing consumption expenditure. Note that a more intuitive scenario would where they are complements, i.e. those who have larger asset pool can afford to take more debt.

A comparison across urban and rural households clearly indicate the urban households are on an average more wealthy than their rural counterparts. This is not very surprising. An interesting feature that Subramanian and Jayaraj (2006) found is that within rural households, those who cultivated typically held more assets than those who do not. This feature can be linked to the broader perspective of the construction of asset portfolios. Indian households typically prefer physical assets than financial assets. There can be a behavioral question as to why do they prefer that? The first simple model of consumption smoothing can not really answer that question, because the consumption smoothing mechanism only requires access to assets. It does not really matter whether it is productive or not. One can conjecture that it might have partly to do with the fact that most Indian households are very close to the borrowing constraints and hence separation of productive and non-productive assets in the portfolio is difficult. Another factor is that people enjoy not only the value of the asset but also the consumption stream (from housing) of it. Gold as an unproductive asset is interesting because the consumption value is also not tangible.

Lahoti et al (2011) (see Ref. [6]) did an analysis on individual asset holding patterns in the Karnataka. They have used data from the Karnataka Household Asset Survey 2010-2011 (KHAS)[2] which has collected data on the asset holdings of individuals (more than 4000 households) from randomly selected 8 districts of the state spread across four regions of Karnataka (Northern Maidan, Southern Maidan, Malnad and Coastal). This particular survey included jewelry as a component of the household asset portfolio, which provides a very important perspective for our purpose. It is clearly seen that there is a gender-gap in holding of productive assets like land, in favor of men over women. Quite intuitively, a reverse gender-gap arises for holding of unproductive[3] assets like gold. This finding is robust across urban as well as rural households.

This is some basic indicative literature. For a complementary perspective and description, interested readers can refer to Ref. [2], [4], [5] and references therein.

At a general level, there are some regularities in wealth and income distributions that apparently hold true across countries and time. The most famous example is the Pareto principle or the 80-20 rule, which states that 20 per cent of the people in an economy holds 80 per cent of wealth. Sinha (2006) (see Ref. [7]) analyzed the data for the richest Indians from 2002-2004 and showed that the wealth distribution of the rich follows a power-law and has a Pareto exponent value in the range of 0.81 and 0.92.[4] On the other hand, the income distribution of the rich is log-normally distributed with a power-law tail and the Pareto exponent approximately equal to 1.5. The fact that the Pareto exponent value for the wealth distribution is lower than the Pareto exponent for the income distribution indicates a greater inequality in the distribution of wealth relative to the distribution of income. This particular feature is actually known in the economics literature. Chakrabarti

[2]This was conducted by the Indian Institute of Management, Bengaluru.

[3]In the sense of economic production process.

[4]We should also note here that there was a long lasting debate about whether the wealth/income distribution actually follows a power law or a log-normal distribution. For a finite range, it is often impossible to statistically differentiate the two. Interested readers can refer to Ref. [10] for a detailed discussion on this issue and a summary of empirical facts.

et al. (2017) (see Ref. [12]) analyzed the data for consumption inequality and showed that the inequality is actually even lesser than that of income. A general observation is that the income and wealth distribution of the bulk of the population of a country (close to 90 percent) exhibits a gamma distribution while the distribution of wealth for the richer sections of the society follows a power-law decay. Interestingly, Gandhi and Walton (2012) (see Ref. [3]) observed that nearly half of the billionaires in India (20 out of 46) derive their wealth from "rent – thick" sectors such as real estate, infrastructure, construction, port sectors, media, cement and mining.

1.2 Plan of the report

In the remaining chapters, we quantitatively analyze the economic and financial nature of gold as an asset. In the next chapter, we will analyze the price and return dynamics of gold in the world market. In the final chapter, we analyze a general equilibrium model for gold accumulation across generations. Finally, we summarize the finding and conclude.

2

Time series properties of the gold price series

In this chapter, I explore some basic time-series properties of the gold price and returns. I denote gold price series as p_t and return by first difference of log prices, i.e.

$$r_t = \frac{\log(p_t)}{\log(p_{t-1})}. \tag{2.1}$$

I have collected time series data on gold prices (Gold Fixing Price 10:30 A.M. (London time) in London Bullion Market, based in U.S. Dollars) from

`https://fred.stlouisfed.org/release?rid=256`

The tools used here are fairly standard and are discussed in graduate text-books. So I am not discussing the background material here. There are many lucid expositions of these tools and techniques; interested readers may refer to Ref. [11] which has a detailed discussion of all material.

2.1 Visual depiction of price and return data

The top panel of Fig. 2.1 shows the evolution of the gold price from 01/04/1968 to 12/03/2018. The bottom panel converts it into return $r(t)$. A feature that emerges from the return series is that volatility seems to cluster around certain time points. This is a commonly known phenomenon for financial time-series, which goes by the name of *volatility clustering.*

In order to examine the nature of clustering, we compute the autocorrelation function (ACF) of the return series and the squared return series. The results are presented in Fig. 2.2. The top panel shows that the *acf* in return is essentially zero for all lags; whereas the *acf* for the squared return decays at a much slower rate.

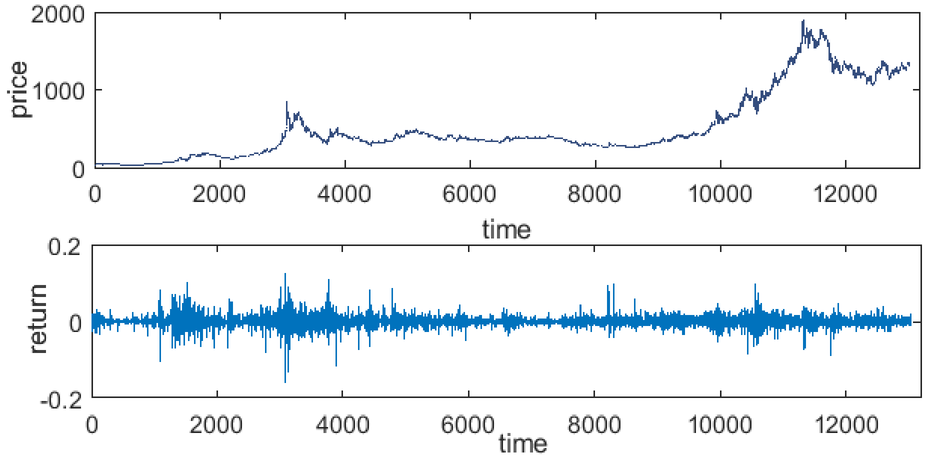

Figure 2.1: Price data (top panel) and return data (bottom panel). The log return series clearly indicates volatility clustering.

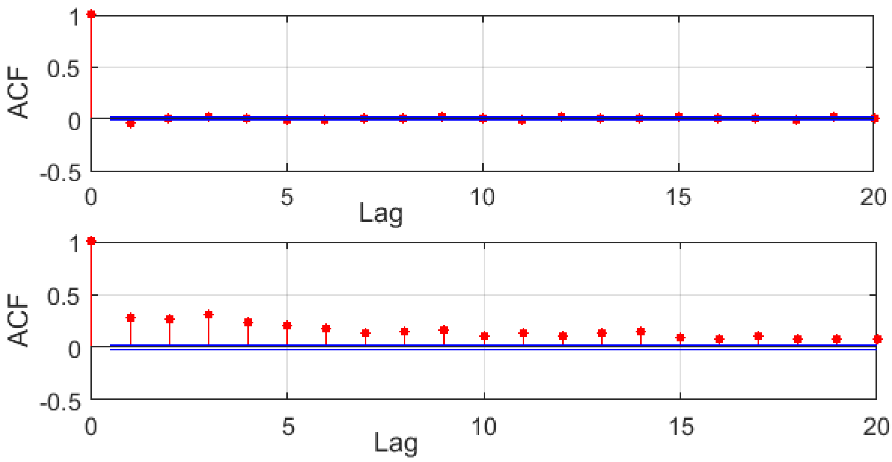

Figure 2.2: Analyzing the autocorrelation function (ACF). The return series has zero autocorrelation at all lags. However, the squared series has positive and significant autocorrelation till lag 20. This long ACF is a feature of many financial time series.

2.2 Modeling volatility

To model the dynamics of volatility, we use three variants of GARCH model. A standard GARCH (p, q) model is given by the following equations:

$$\begin{aligned} r_t &= \epsilon_t, \\ \epsilon_t|\psi_{t-1} &\sim N(0, \sigma_t^2) \quad \text{where } \psi_{t-1} \text{ is the information set,} \\ \sigma_t^2 &= \omega + \sum_i^p \alpha_i \epsilon_{t-i}^2 + \sum_j^q \beta_j \sigma_{t-j}^2. \end{aligned} \tag{2.2}$$

I ran a GARCH(1,1) model on the data. Results are presented below.

```
GARCH(1,1) Conditional Variance Model:
    ----------------------------------------
    Conditional Probability Distribution: Gaussian

                                     Standard          t
     Parameter          Value          Error        Statistic
    -----------    -----------    ------------    -----------
     Constant      9.86673e-07    1.13118e-07        8.7225
     GARCH{1}         0.896862      0.0016029       559.526
      ARCH{1}         0.103138     0.00191193       53.9443
```

Nelson and Cao (1991) introduced EGARCH version which allows for exponential weightage. The exponential GARCH (EGARCH) model admits the following equations with modifications on the basic GARCH model:

$$\begin{aligned} r_t &= \epsilon_t, \\ \epsilon_t|\psi_{t-1} &\sim N(0, \sigma_t^2) \quad \text{where } \psi_{t-1} \text{ is the information set,} \\ \log \sigma_t^2 &= \omega + \sum_i^p \gamma_i g(Z_{t-i})^2 + \sum_j^q \beta_j \log \sigma_{t-j}^2, \end{aligned} \tag{2.3}$$

where the function $g(x) = \theta x + \lambda(|x| - E(|x|))$ (θ and λ are parameters) and Z is a standard normal variable. The construction of the function $g(.)$ is such that it allows for different effects of sign and magnitude. The results are presented below.

```
    EGARCH(1,1) Conditional Variance Model:
    --------------------------------------
```

```
    Conditional Probability Distribution: Gaussian

                                  Standard          t
     Parameter       Value          Error       Statistic
    -----------   -----------   ------------   -----------
     Constant      -0.180051     0.00676116      -26.6303
     GARCH{1}       0.977851    0.000710922       1375.47
      ARCH{1}       0.250753     0.00310781       80.6846
  Leverage{1}      0.0245112     0.00191809        12.779
```

Finally, we also model volatility using Glosten-Jagannathan-Runkle GARCH (GJR-GARCH). We take the innovation term $\epsilon_t = \sigma_t x_t$ where x_t is independent and identically distributed error term; and, we incorporate a step function I_t to allow for asymmetry in the ARCH process for volatility. The new volatility equation for GJR-GARCH(1,1) looks as follows:

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2 + \gamma\epsilon_{t-1}^2.I_{\epsilon_{t-1}<0}. \quad (2.4)$$

Thus if the realization of ϵ is negative, then there will be a positive impact on the volatility. This allows for inclusion of asymmetric impact of positive and negative returns on volatility. This kind of modeling is motivated by empirical properties of the financial market. Estimation results are presented below.

```
GJR(1,1) Conditional Variance Model:
    --------------------------------------
    Conditional Probability Distribution: Gaussian

                                  Standard          t
     Parameter       Value          Error       Statistic
    -----------   -----------   ------------   -----------
     Constant    9.31708e-07    1.11833e-07       8.33125
     GARCH{1}       0.900322     0.00150577       597.915
      ARCH{1}       0.116689     0.00247857       47.0794
  Leverage{1}     -0.0340232     0.00263994      -12.8879
```

2.3 Spectral analysis

Here the essential idea is to decompose the return series $r(t)$ into a combination of sinusoidal curves. Thus we translate the time series information into frequency domain information. Given the series $r(t)$, I execute a discrete Fourier transform (following the notation in Matlab documentation) as follows:

$$Y(k) = \sum_{j}^{n} r(j) W_n^{(j-1)(k-1)} \tag{2.5}$$

where

$$W_n = exp(-2\pi i/n). \tag{2.6}$$

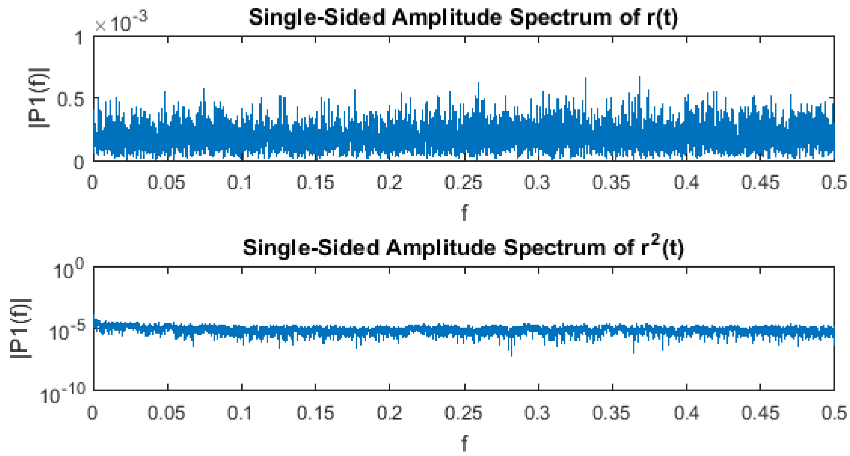

Figure 2.3: Single-Sided Amplitude Spectrum of the return series.

The results are presented in Fig. 2.3 for $r(t)$ and $r^2(t)$. The results do not indicate any particular periodic movement.

3

General equilibrium modeling of wealth accumulation

In this section, I develop a model for analyzing the mechanism of gold accumulation across generations.[1] I consider an economy where gold is the only asset that is used for trading purpose as well as for the purpose of bequests. Below, I first discuss the modeling approach and provide three variations of the model which generates non-trivial distribution of assets across households. Finally, I will discuss potential extensions of the framework.

3.1 Modeling approach: General equilibrium

The main motivation and theoretical structure for this type of modeling have been discussed in details in [9] and [10]. In order to avoid repetition, I am not discussing them here. Interested readers should consult these two references and references contained therein.

There are N dynasties denoted by $i \in N$. Each dynasty has an endowment of gold g_{i0} at the beginning of time i.e. $t = 0$. Each dynasty is populated by one person at every point of time who lives for one period.[2] At the beginning of each period, the i-th dynasty ($i \in N$) gets an endowment of a tradable good $Q_i > 0$.

We assume that

$$Q_i \sim uni[0, 1] \tag{3.1}$$

and in general,

$$Q_i \neq Q_j \tag{3.2}$$

[1] I gratefully acknowledge research assistance provided by Sanjay Moorjani for numerically solving the models discussed in this chapter.

[2] We can easily make the model two-periods with the usual interpretation of overlapping generations. However, for the current exercise, one-period construction suffices.

for all $\{i, j\}$.

The assumption of uniform distribution for the production endowment is not restrictive. The model we propose will work for any positive value fo endowments. However, we have to be careful while using distributions like Normal for example, as theoretically that kind of distributions will include negative values as well.

3.1.1 Bilateral matching

Once the dynasties get their endowments they are bilaterally matched with trading partners. We assume that the endowments are differentiated across the place of origin, i.e. different dynasties produce different types of output. Given that the utility functions are defined over all types of goods, there would be an incentive for all dynasties to trade.

In principle, we can consider a common market place where all representatives of all dynasties meet at every point of time and carry out trade. However, here we consider a more realistic scenario of only two randomly chosen agents (from two different dynasties) meeting with each other and carry out trade. Thus at every point of time, each dynasty has a probability of $1/N$ of participation in the trading activity.

3.2 Trading mechanism

Trading occurs across two traders over two goods, and gold functions as a method to settle payments. Households also derive utility from bequesting gold to the next generations.

We accommodate this feature by using *money in the utility function* formalism. This kind of modeling is also related to the *warm-glow* utility functions, where the household derives utility from bequests.

We impose a competitive trading mechanism. Prices are set such that the markets for both goods clear. Then by Walras' law, the market for gold also clears.

3.3 Numerical recipe for solving $f(x) = 0$

For the competitive model, the market clearing equations turn out to be nonlinear in prices. Hence, solving for the market clearing prices become analytically almost impossible (except under some restrictive conditions). So we resort to numerical methods to find out the equilibrium prices.

Suppose, we denote the excess demand of a commodity as a function of the price vector p:

$$x = x(p). \tag{3.3}$$

Then the equilibrium is achieved where all commodity markets clear. We succintly write it as (in vector notation)

$$f(x) = 0. \tag{3.4}$$

To solve the set of equations, we follow Broyden's method (generalization of the gradient-descent method). Here, we encounter a problem. Ideally, there is one price vector which clears the markets.[3] However, given the set of market clearing equations we cannot guarantee that there is an unique solution. Th bigger problem is that some the solutions could be complex numbers, which makes little economic sense.

Essentially, the problem of convergence and complex solutions cannot be avoided in this set up. Hence, although the models are well-defined the solutions often are not. Numerically, we attempt to solve the equations for each trade. But for repeated trading for a large number of iterations (around 10^3 trades for each round and averaged over 10^3 rounds), sometimes the algorithm breaks midway as it either fails to converge or produces complex solutions for prices.[4]

With trial and error, we have identified several parameter configurations, for which the solution can be found numerically. Below, we first describe the details of the model and then provide the solutions. In particular, we have three variants of the model, each with different interpretation of the utility function.

3.4 Model I

It is a model with binary trading. Time is discrete. At a generic time point t, two agents (1 and 2) are chosen with replacement.

For agent 1, the utility function and the budget constraint is given as follows:

$$U_1 = \left(a_1 x_1^r + a_2 x_2^r + a_3 g_1^r\right)^{\frac{1}{r}} \quad \text{where } a_1 + a_2 + a_3 = 1 \tag{3.5}$$

$$BC_1 : p_1 x_1 + p_2 x_2 + g_1 \leq p_1 Q_1 + G_1 \tag{3.6}$$

$$\tag{3.7}$$

For agent 2, the utility function and the budget constraint is given as follows:

$$U_2 = \left(b_1 y_1^r + b_2 y_2^r + b_3 g_2^r\right)^{\frac{1}{r}}, \text{where } b_1 + b_2 + b_3 = 1 \tag{3.8}$$

$$BC_2 : p_1 y_1 + p_2 y_2 + g_2 \leq p_2 Q_2 + G_2 \tag{3.9}$$

[3] We know this from basic general equilibrium theory. See for example, Mascollel, Whinston and Green (2002). We can use Brower's fixed point theorem to show existence of a solution.

[4] We have provided codes for both Matlab and python in the appendix of this report. This problem is not programming language specific.

3.4.1 First order conditions

We first set up the optimization problem using a Lagrange multiplier:

$$L_1 = (a_1x_1^r + a_2x_2^r + a_3g_1^r)^{\frac{1}{r}} + \lambda_1(p_1Q_1 + G_1 - p_1x_1 - p_2x_2 - g_1) \quad (3.10)$$

$$L_2 = (b_1y_1^r + b_2y_2^r + b_3g_2^r)^{\frac{1}{r}} + \lambda_2(p_2Q_2 + G_2 - p_1y_1 - p_2y_2 - g_2). \quad (3.11)$$

Below, we list the first order conditions:

$$\frac{\partial L_1}{\partial x_1} : (a_1x_1^r + a_2x_2^r + a_3g_1^r)^{\frac{1}{r}-1} a_1x_1^{r-1} - \lambda_1p_1 = 0 \quad (3.12)$$

$$\frac{\partial L_1}{\partial x_2} : (a_1x_1^r + a_2x_2^r + a_3g_1^r)^{\frac{1}{r}-1} a_2x_2^{r-1} - \lambda_1p_2 = 0 \quad (3.13)$$

$$\frac{\partial L_1}{\partial g_1} : (a_1x_1^r + a_2x_2^r + a_3g_1^r)^{\frac{1}{r}-1} a_3g_1^{r-1} - \lambda_1 = 0 \quad (3.14)$$

Hence,
$$\frac{a_1x_1^{r-1}}{p_1} = \frac{a_2x_2^{r-1}}{p_2} = a_3g_1^{r-1} \quad (3.15)$$

Similarly,
$$\frac{b_1y_1^{r-1}}{p_1} = \frac{b_2y_2^{r-1}}{p_2} = b_3g_2^{r-1}. \quad (3.16)$$

3.4.2 Demand functions

After solving the FOCs, we get the following demand functions (for both agents, both goods as well as for gold):

$$x_1^* = \frac{G_1+p_1Q_1}{p_1+p_2\left(\frac{p_2a_1}{p_1a_2}\right)^{\frac{1}{r-1}}+\left(\frac{a_1}{p_1a_3}\right)^{\frac{1}{r-1}}} \quad (3.17)$$

$$x_2^* = \frac{G_1+p_1Q_1}{p_2+p_1\left(\frac{p_1a_2}{p_2a_1}\right)^{\frac{1}{r-1}}+\left(\frac{a_2}{p_2a_3}\right)^{\frac{1}{r-1}}} \quad (3.18)$$

$$y_1^* = \frac{G_2+p_2Q_2}{p_1+p_2\left(\frac{p_2b_1}{p_1b_2}\right)^{\frac{1}{r-1}}+\left(\frac{b_1}{p_1b_3}\right)^{\frac{1}{r-1}}} \quad (3.19)$$

$$y_2^* = \frac{G_2+p_2Q_2}{p_2+p_1\left(\frac{p_1b_2}{p_2b_1}\right)^{\frac{1}{r-1}}+\left(\frac{b_2}{p_2b_3}\right)^{\frac{1}{r-1}}} \quad (3.20)$$

$$g_1^* = \frac{G_1+p_1Q_1}{p_1\left(\frac{p_1a_3}{a_1}\right)^{\frac{1}{r-1}}+p_2\left(\frac{p_2a_3}{a_2}\right)^{\frac{1}{r-1}}+1} \quad (3.21)$$

$$g_2^* = \frac{G_2+p_2Q_2}{p_1\left(\frac{p_1b_3}{b_1}\right)^{\frac{1}{r-1}}+p_2\left(\frac{p_2b_3}{b_2}\right)^{\frac{1}{r-1}}+1} \quad (3.22)$$

3.4.3 Market Clearing equations

Combining the first order conditions, we can write the market clearing equations as follows.

$$\frac{G_1 + p_1 Q_1}{p_1 + p_2\left(\frac{p_2 a_1}{p_1 a_2}\right)^{\frac{1}{r-1}} + \left(\frac{a_1}{p_1 a_3}\right)^{\frac{1}{r-1}}} + \frac{G_2 + p_2 Q_2}{p_1 + p_2\left(\frac{p_2 b_1}{p_1 b_2}\right)^{\frac{1}{r-1}} + \left(\frac{b_1}{p_1 b_3}\right)^{\frac{1}{r-1}}} = Q_1 \quad (3.23)$$

$$\frac{G_1 + p_1 Q_1}{p_2 + p_1\left(\frac{p_1 a_2}{p_2 a_1}\right)^{\frac{1}{r-1}} + \left(\frac{a_2}{p_2 a_3}\right)^{\frac{1}{r-1}}} + \frac{G_2 + p_2 Q_2}{p_2 + p_1\left(\frac{p_1 b_2}{p_2 b_1}\right)^{\frac{1}{r-1}} + \left(\frac{b_2}{p_2 b_3}\right)^{\frac{1}{r-1}}} = Q_2 \quad (3.24)$$

3.4.4 Distribution of gold across dynasties

We are now in a position to analyze the distribution of assets across dynasties. Note that the same model also generates the dynamics of gold-holding across generations within a particular dynasty.

In the following figures, we provide the distribution of wealth across dynasties along with an exponential distribution. We see that as the relative weight $a3$ increases, the mode of the distribution shifts to the right, reducing inequality.

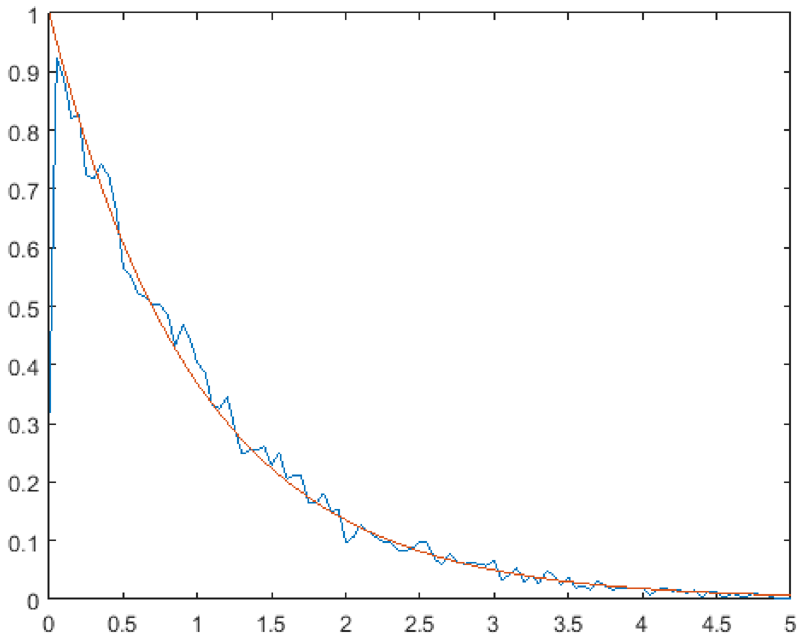

Figure 3.1: Simulation Results for Model I: a3=0.02, sigma=0. The red line indicates an exponential distribution. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

3.5 Model II

Similar to the earlier model, it is a model with binary trading. Time is discrete. At a generic time point t, two agents (1 and 2) are chosen with replacement.

For agent 1, the utility function and the budget constraint is given as follows:

$$U_1 = x_1^{a_1} x_2^{a_2} g_1^{a_3}, \text{where } a_1 + a_2 + a_3 = 1 \tag{3.25}$$
$$BC_1 \quad : p_1 x_1 + p_2 x_2 + g_1 \leq p_1 Q_1 + G_1 \tag{3.26}$$
$$\tag{3.27}$$

For agent 2, the utility function and the budget constraint is given as follows:

$$U_2 = x_1^{b_1} x_2^{b_2} g_2^{b_3}, \text{where } b_1 + b_2 + b_3 = 1 \tag{3.28}$$
$$BC_2 \quad : p_1 y_1 + p_2 y_2 + g_2 \leq p_2 Q_2 + G_2 \tag{3.29}$$

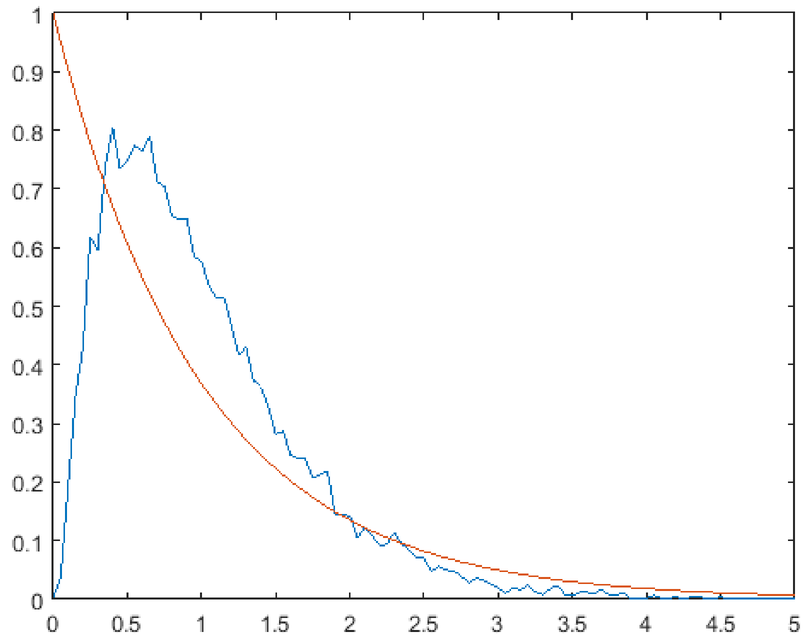

Figure 3.2: Simulation Results for Model I: a3=0.3, sigma=0. The red line indicates an exponential distribution. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

3.5.1 First order conditions

We first set up the optimization problem using a Lagrange multiplier:

$$L_1 = x_1^{a_1} x_2^{a_2} m_1^{a_3} + \lambda_1(p_1 Q_1 + G_1 - p_1 x_1 - p_2 x_2 - g_1) \tag{3.30}$$

$$L_2 = x_1^{b_1} x_2^{b_2} m_2^{b_3} + \lambda_2(p_2 Q_2 + G_2 - p_1 y_1 - p_2 y_2 - g_2) \tag{3.31}$$

Below, we list the first order conditions:

$$\frac{\partial L_1}{\partial x_1} : \quad a_1 x_1^{a_1 - 1} x_2^{a_2} g_1^{a_3} - \lambda_1 p_1 = 0 \tag{3.32}$$

$$\frac{\partial L_1}{\partial x_2} : \quad a_2 x_1^{a_1} x_2^{a_2 - 1} g_1^{a_3} - \lambda_1 p_2 = 0 \tag{3.33}$$

$$\frac{\partial L_1}{\partial m_1} : \quad a_3 x_1^{a_1} x_2^{a_2} g_1^{a_3 - 1} - \lambda_1 = 0. \tag{3.34}$$

Hence,

$$\frac{a_1}{x_1 p_1} = \frac{a_2}{x_2 p_2} = \frac{a_3}{g_1}.$$

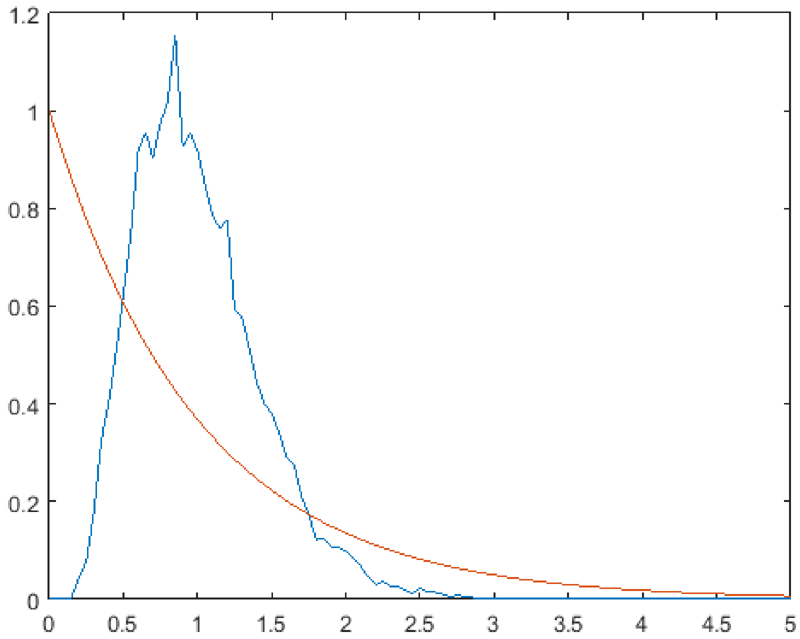

Figure 3.3: Simulation Results for Model I: a3=0.6, sigma=0. The red line indicates an exponential distribution. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

Similarly,

$$\frac{b_1}{y_1 p_1} = \frac{b_2}{y_2 p_2} = \frac{b_3}{g_2}.$$

3.5.2 Demand functions

After solving the FOCs, we get the following demand functions (for both agents, both goods as well as for gold):

$$x_1^* = a_1 \tfrac{G_1 + p_1 Q_1}{p_1} \tag{3.35}$$

$$x_2^* = a_2 \tfrac{G_1 + p_1 Q_1}{p_2} \tag{3.36}$$

$$y_1^* = b_1 \tfrac{G_2 + p_2 Q_2}{p_1} \tag{3.37}$$

$$y_2^* = b_2 \tfrac{G_2 + p_2 Q_2}{p_2} \tag{3.38}$$

$$g_1^* = a_3 (G_1 + p_1 Q_1) \tag{3.39}$$

$$g_2^* = b_3 (G_2 + p_2 Q_2) \tag{3.40}$$

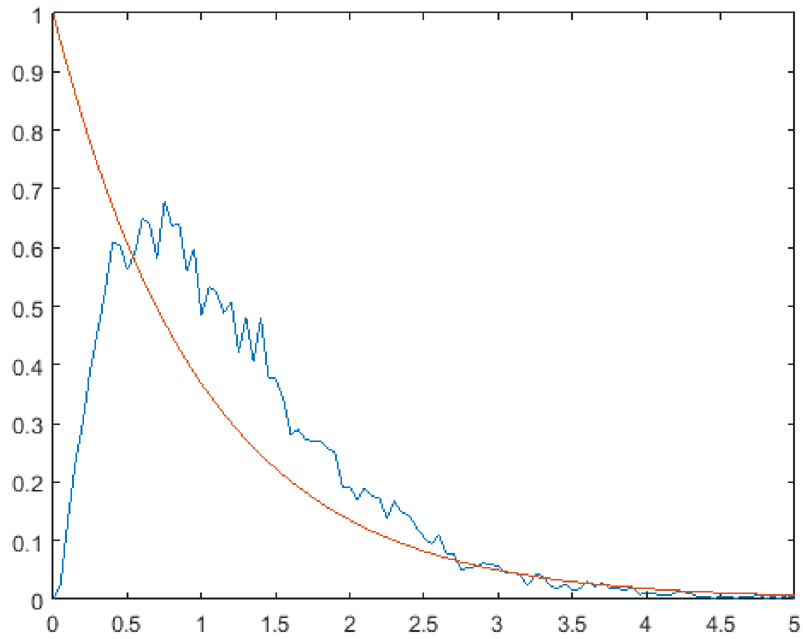

Figure 3.4: Simulation Results for Model I: a3=0.02, sigma=0.1. The red line indicates an exponential distribution. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

3.5.3 Market Clearing equations

Combining the first order conditions, we can write the market clearing equations as follows.

$$a_1 \frac{G_1 + p_1 Q_1}{p_1} + b_1 \frac{G_2 + p_2 Q_2}{p_1} = Q_1 \tag{3.41}$$
$$a_2 \frac{G_1 + p_1 Q_1}{p_2} + b_2 \frac{G_2 + p_2 Q_2}{p_2} = Q_2 \tag{3.42}$$

3.5.4 Distribution of gold across dynasties

Similar to model I, here also we can analyze the distribution of assets across dynasties. The same model also generates the dynamics of gold-holding across generations within a particular dynasty. We ignore the wealth accumulation dynamics here.

In the following figures, we provide the distribution of wealth across dynasties along with an exponential distribution.

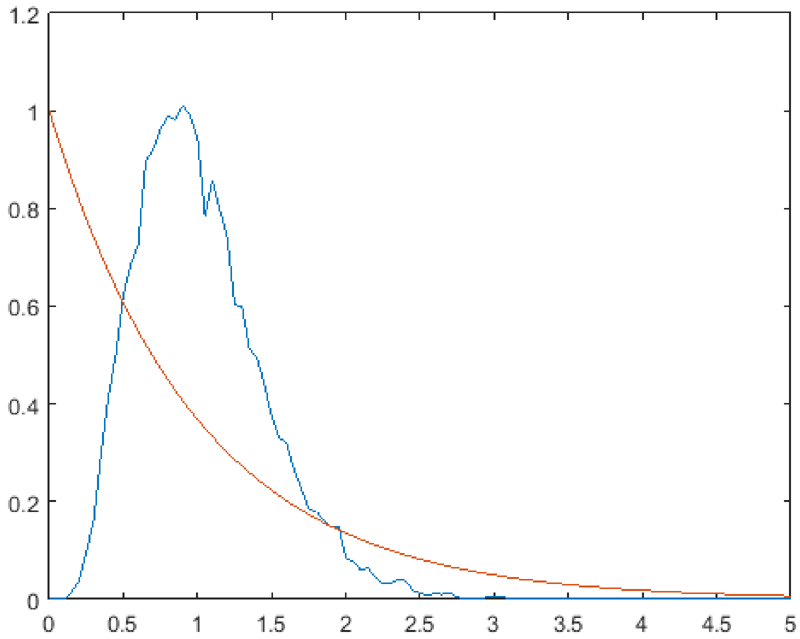

Figure 3.5: Simulation Results for Model I: a3=0.3, sigma=0.1. The red line indicates an exponential distribution. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

3.6 Model III

Similar to model I, it is a model with binary trading. Time is discrete. At a generic time point t, two agents (1 and 2) are chosen with replacement.

For agent 1, the utility function and the budget constraint is given as follows:

$$U_1 = \mu_1 x_1^{a_1} x_2^{a_2} + (1-\mu_1) g_1^{\gamma}, \text{ where } a_1 + a_2 = 1 \tag{3.43}$$

$$BC_1 : p_1 x_1 + p_2 x_2 + g_1 \leq p_1 Q_1 + G_1 \tag{3.44}$$

$$\tag{3.45}$$

For agent 2, the utility function and the budget constraint is given as follows:

$$U_2 = \mu_2 y_1^{b_1} y_2^{b_2} + (1-\mu_2) g_2^{\gamma}, \text{ where } b_1 + b_2 = 1 \tag{3.46}$$

$$BC_2 : p_1 y_1 + p_2 y_2 + g_2 \leq p_2 Q_2 + G_2 \tag{3.47}$$

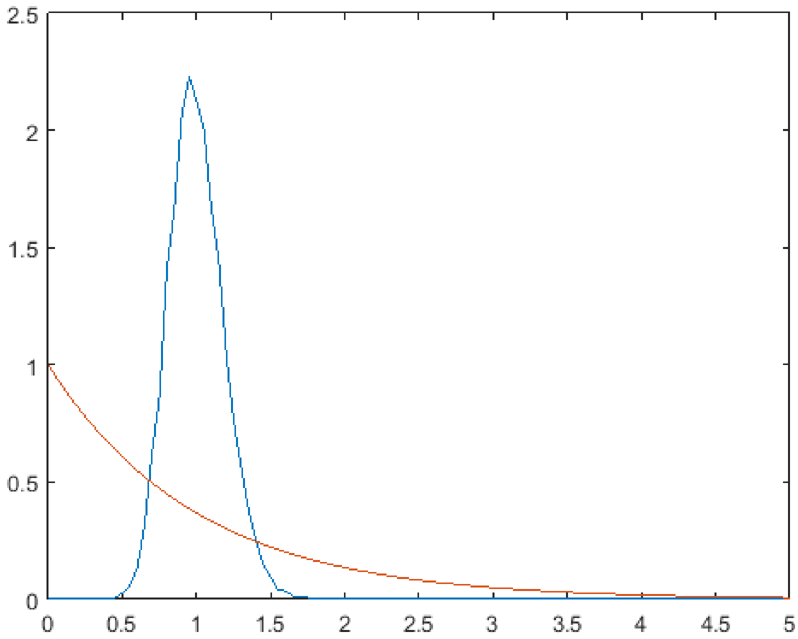

Figure 3.6: Simulation Results for Model I: a3=0.6, sigma=0.1. The red line indicates an exponential distribution. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

3.6.1 First order conditions

3.6.2 First order conditions

We first set up the optimization problem using a Lagrange multiplier:

$$L_1 = \mu_1 x_1^{a_1} x_2^{a_2} + (1-\mu_1) g_1^{\gamma} + \lambda_1 (p_1 Q_1 + G_1 - p_1 x_1 - p_2 x_2 - g_1) \quad (3.48)$$
$$L_2 = \mu_2 y_1^{b_1} y_2^{b_2} + (1-\mu_2) g_2^{\gamma} + \lambda_2 (p_2 Q_2 + G_2 - p_1 y_1 - p_2 y_2 - g_2) \quad (3.49)$$

Below, we list the first order conditions:

$$\frac{\partial L_1}{\partial x_1} : \frac{\mu_1 a_1 x_1^{a_1} x_2^{a_2}}{x_1} - \lambda_1 p_1 = 0 \quad (3.50)$$
$$\frac{\partial L_1}{\partial x_2} : \frac{\mu_1 a_2 x_1^{a_1} x_2^{a_2}}{x_2} - \lambda_1 p_2 = 0 \quad (3.51)$$
$$\frac{\partial L_1}{\partial g_1} : \gamma(1-\mu_1) g_1^{\gamma-1} - \lambda_1 = 0. \quad (3.52)$$

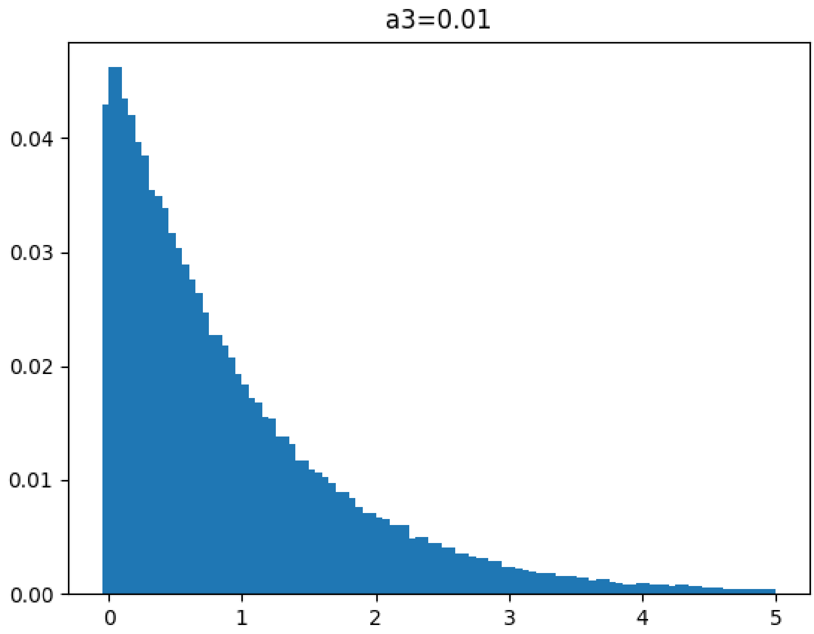

Figure 3.7: Simulation Results for Model II: a3=0.01. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

3.6.3 Demand functions

After solving the FOCs, we get the following demand functions (for both agents, both goods as well as for gold). In this case we cannot explicitly solve for the demand functions. However, we know that at optimal, the following equations hold:

$$\begin{aligned} \frac{\mu_1 a_1 x_1^{a_1-1} x_2^{a_2}}{p_1} &= \frac{\mu_1 a_2 x_1^{a_1} x_2^{a_2-1}}{p_2} \\ &= \gamma(1-\mu_1) g_1^{\gamma-1}. \end{aligned} \tag{3.53}$$

This equation is nothing but MRS = price ratio. Hence, the basic intuitions of optimization suffices to see why it appears. Similarly, for the second agent,

$$\begin{aligned} \frac{\mu_2 b_1 y_1^{b_1-1} y_2^{b_2}}{p_1} &= \frac{\mu_2 b_2 y_1^{b_1} y_2^{b_2-1}}{p_2} \\ &= \gamma(1-\mu_2) g_2^{\gamma-1}. \end{aligned} \tag{3.54}$$

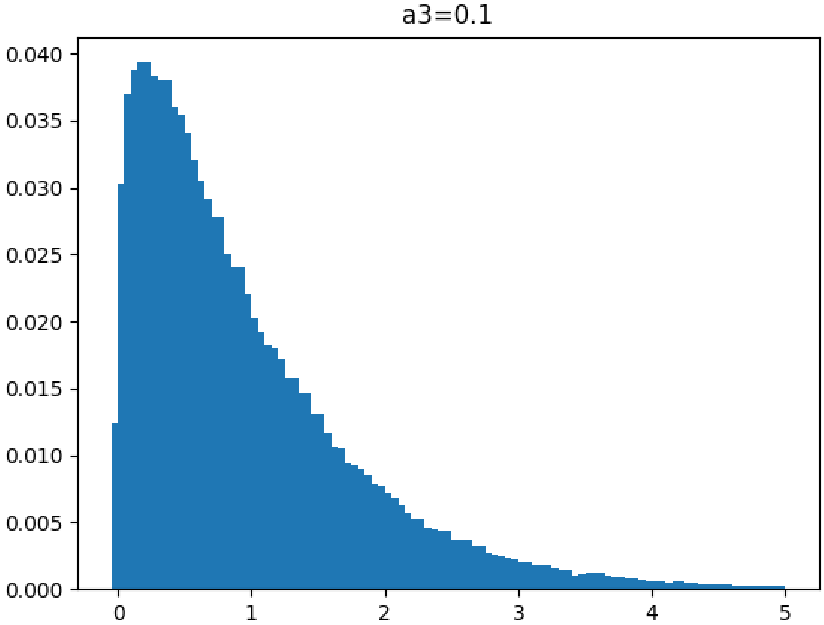

Figure 3.8: Simulation Results for Model II: a3=0.1. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

3.6.4 Numerically solved system of equations

Since this model cannot be solved explicitly, I collect all the relevant equations here. The equations for first order conditions as well as for market clearance are shown below.

$$\mu_1 a_1 {x_1^*}^{a_1-1} {x_2^*}^{a_2} p_2 = \mu_1 a_2 {x_1^*}^{a_1} {x_2^*}^{a_2-1} p_1 \tag{3.55}$$
$$\mu_1 a_1 {x_1^*}^{a_1-1} {x_2^*}^{a_2} = \gamma(1-\mu_1) {g_1^*}^{\gamma-1} p_1 \tag{3.56}$$
$$\mu_1 b_1 {y_1^*}^{b_1-1} {y_2^*}^{b_2} p_2 = \mu_1 b_2 {y_1^*}^{b_1} {y_2^*}^{b_2-1} p_1 \tag{3.57}$$
$$\mu_1 b_1 {y_1^*}^{b_1-1} {y_2^*}^{b_2} = \gamma(1-\mu_1) {g_2^*}^{\gamma-1} p_1 \tag{3.58}$$
$$p_1 x_1^* + p_2 x_2^* + g_1^* = p_1 Q_1 + G_1 \tag{3.59}$$
$$p_1 y_1^* + p_2 y_2^* + g_2^* = p_2 Q_2 + G_2 \tag{3.60}$$
$$x_1^* + y_1^* = Q_1 \tag{3.61}$$
$$x_2^* + y_2^* = Q_2 \tag{3.62}$$

where, $x_1^*, x_2^*, y_1^*, y_2^*, g_1^*, g_2^*, p_1$ and p_2 are the unknowns.

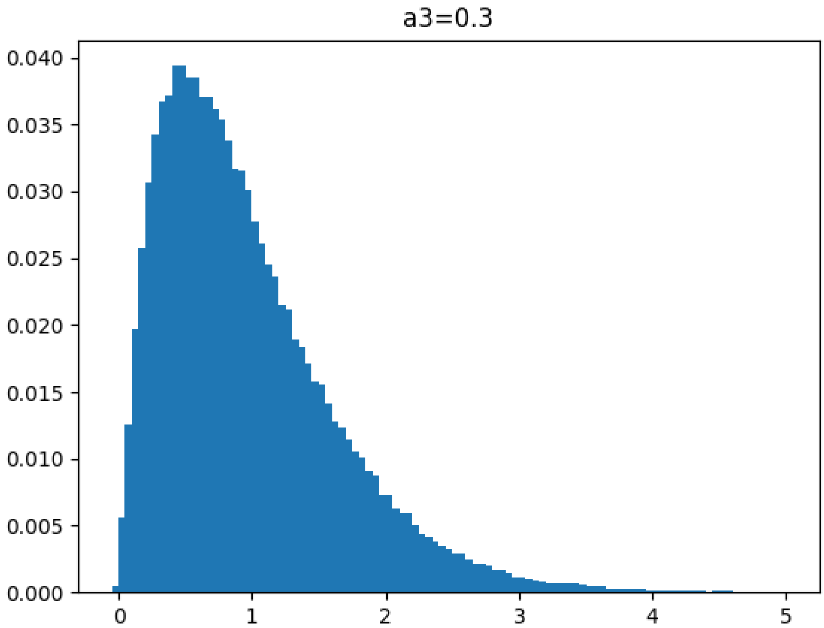

Figure 3.9: Simulation Results for Model II: a3=0.3. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

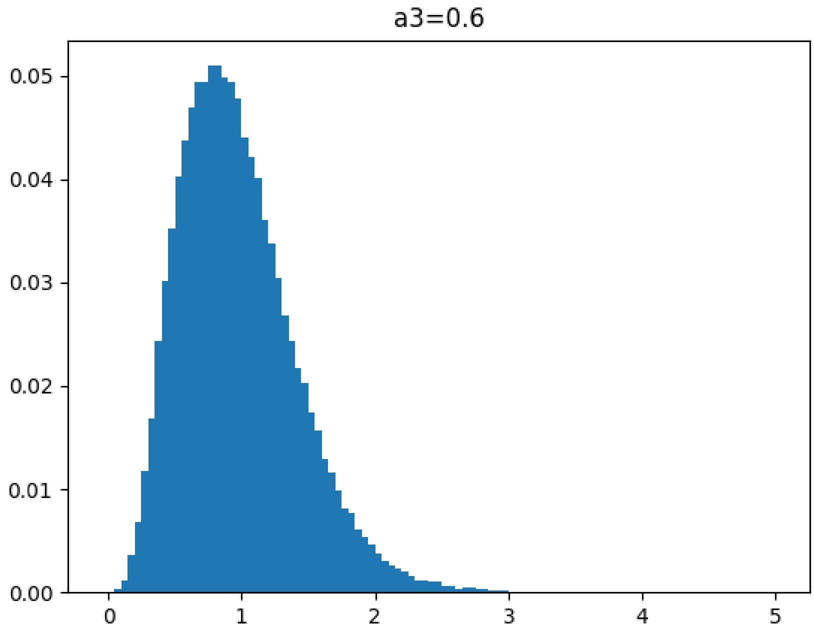

Figure 3.10: Simulation Results for Model II: a3=0.6. The x-axis denotes gold holding g, y-axis denotes the probability density function $P(g)$.

4

Summary and conclusion

In this report, I have analyzed gold price dynamics as well as the distribution of gold in an economy with heterogeneous households. We see that gold return series has time-series properties very similar to those of other financial assets.

Then I provide a model with overlapping generations mechanism that allows me to model wealth distribution through *warm-glow* utility functions. This model has two properties: it generates time-series properties for individual households and, it also generates distribution of assets across households in the steady state.

One problem with the framework is that often prices do not converge to the equilibrium values. In most of the cases (except under very restrictive assumptions) these models are not analytically solvable. Hence, we resort to numerical solutions. However, even in simple cases, the equations are so nonlinear that the prices do not converge with standard numerical root-finding algorithms.

In future, an interesting direction would to solve the model for a general specification of the utility function and stochastic endowments. If one can find a robust method for solving that, then we can the model as a laboratory to find out the effect of preferences and production structure on inequality.

5

Appendix

5.1 Matlab codes

5.1.1 Binary trade mechanism

```
%Implement computable general equilibrium trade
%Generalized kinetic exchange model\\
%Numerical solution to KWEM general equilibrium method\\

%Written by Anindya S. Chakrabarti, 21/2/2018\\

%Utility function: u=(a_1x_1^r+ b_1x_2^r+ a_3m^r)^(1/r)

clc;
clear all;

n=2;
money_mat=[1 1];

%List of paramters
global M1 M2 Q1 Q2 a1 a2 a3 b1 b2 b3 r

%Exogenous parameters
Q1=1;
Q2=1;

%Preference parameters
r=.001;
a3=.2;b3=a3;

i=1;j=2;
M1=money_mat(1);M2=money_mat(2);
```

```
a1=.490;
a2=1-a3-a1;
b1=a1;b2=a2;

TheoryP=ones(n,1);
TheoryP(1)=(a1/a3)*(M1+M2)/Q1;TheoryP(2)=(a2/a3)*(M1+M2)/Q2;
fprintf('Initial: P1,P2=%.4f,%.4f\n',TheoryP(1),TheoryP(2));

P=ones(n,1);
%options=optimset('Algorithm','Levenberg-Marquardt','MaxIter',500);%,'Display',
options=optimset('Algorithm','trust-region-reflective','MaxIter',500);

[P,~,exitflag] = fsolve(@focfn,ones(size(P)),options);

P1=P(1);P2=P(2);

m1=(M1+P1*Q1)/(1+P1*(P1*a3/a1)^(1/(r-1))+P2*((P2*a3)/a2)^(1/(r-1)));
m2=(M2+P2*Q2)/(1+P1*(P1*b3/b1)^(1/(r-1))+P2*((P2*b3)/b2)^(1/(r-1)));

fprintf('M1,M2=%.4f,%.4f\n',M1,M2);

fprintf('m1,m2=%.4f,%.4f\n',m1,m2);

fprintf('P1,P2=%.4f,%.4f\n',P1,P2);
```

5.1.2 Binary trades with multiple agents

```
%Implement computable general equilibrium trade
%Generalized kinetic exchange model
%Numerical solution to KWEM general equilibrium method

%Written by Anindya S. Chakrabarti, 21/2/2018

%Utility function: u=(a_1x_1^r+ b_1x_2^r+ a_3m^r)^(1/r)

clc;
clear all;

n=2;   %n-ary trade
N=100; %Number of agents
T=500; %Iterations in one instance


unit=0.05; %Unit on x-axis for plotting pdf
```

```
x_axis=0:unit:N;


tic;

money_mat=zeros(N,T);
money_mat(:,1)=ones(N,1);

%List of paramters
global M1 M2 Q1 Q2 a1 a2 a3 b1 b2 b3 r

%Exogenous parameters
Q1=1;
Q2=1;

%Preference parameters
r=.1;
a3=.01;b3=a3;

store_a=ones(T,1).*2;

for t=2:T;

choose_agents=randsample(N,n);
i=choose_agents(1);j=choose_agents(2);

M1=money_mat(i,t-1);M2=money_mat(j,t-1);

a1=unifrnd(0,1-a3);
a2=1-a3-a1;
store_a(t)=min(a1,a2);

b1=a1;%unifrnd(0,1-b3);
b2=a2;%1-b3-b1;



P=ones(n,1);
%options=optimset('Algorithm','Levenberg-Marquardt');%,...
%    'Display','iter')%,'PlotFcn',@optimplotfirstorderopt);
%options=optimset('MaxIter',500);
[P,~,exitflag] = fsolve(@focfn,ones(size(P)));%,options);
```

```
fprintf('min(a1,a2)=%.4f,t=%d \n',min(a1,a2),t);

if (exitflag<0);
    break;
end;
P1=P(1);P2=P(2);

m1=(M1+P1*Q1)/(1+P1*(P1*a3/a1)^(1/(r-1))...
    +P2*((P2*a3)/a2)^(1/(r-1)));
m2=(M2+P2*Q2)/(1+P1*(P1*b3/b1)^(1/(r-1))...
    +P2*((P2*b3)/b2)^(1/(r-1)));

money_mat(:,t)=money_mat(:,t-1);
money_mat(i,t)=m1;money_mat(j,t)=m2;

end;

[y,x]=hist(money_mat(:,t-1),x_axis);

figure;
subplot(1,3,1);
plot(x_axis,y./(sum(y)*unit));
xlim([0 5]);xlabel('w');ylabel('P(w)');
subplot(1,3,2);
plot(store_a(2:t),'-o');xlabel('time');ylabel('min(a1,a2)');
ylim([0 1-a3]);
subplot(1,3,3);
plot(sum(money_mat,1),'r-o');xlabel('time');ylabel('sum_i w_i');
ylim([N-1 N+1]);


toc;
```

5.1.3 Binary trades with multiple agents and averaging

```
%Implement computable general equilibrium trade
%Generalized kinetic exchange model
%Numerical solution to KWEM: general equilibrium method

%Written by Anindya S. Chakrabarti, 26/2/2018

%Utility function: u=(a_1x_1^r+ b_1x_2^r+ a_3m^r)^(1/r)
```

```
clc;
clear all;

n=2;   %n-ary trade
N=100; %Number of agents
T=2000; %Iterations in one instance
MaxInstances=100; %Number of instances
delta=.0; %For numerical purpose we need to adjust the range of
          %a1 and a2 such that min(a1,a2)>delta.

unit=0.05; %Unit on x-axis for plotting pdf
x_axis=0:unit:N;
store_pdf=zeros(MaxInstances,length(x_axis));

tic;
%store_money_mat=zeros(N,T,MaxInstances);

%List of paramters
global M1 M2 Q1 Q2 a1 a2 a3 b1 b2 b3 r

%Preference parameters
r=0.01;
a3=.02; %Do NOT take a3<0.02
b3=a3;

%Production parameter
mu=1;
sigma=0.01;

for inst=1:MaxInstances;

money_mat=zeros(N,T);
money_mat(:,1)=ones(N,1);




for t=2:T;

%Production level
Q1=normrnd(mu,sigma);
Q2=normrnd(mu,sigma);
```

```
choose_agents=randsample(N,n);
i=choose_agents(1);j=choose_agents(2);

M1=money_mat(i,t-1);M2=money_mat(j,t-1);

a1=unifrnd(delta,1-a3-delta);  % NOTICE THE DELTA ADJUSTMENT
a2=1-a3-a1;
b1=a1;%unifrnd(0,1-b3);
b2=a2;%1-b3-b1;



P=ones(n,1);
options=optimset('Algorithm','Levenberg-Marquardt','MaxIter',500);%,...
%    'Display','iter')%,'PlotFcn',@optimplotfirstorderopt);
%options=optimset('MaxIter',500);
[P,~,exitflag] = fsolve(@focfn,ones(size(P)),options);


if (exitflag<0||min(P)<0);
    return;
end;
P1=P(1);P2=P(2);

m1=(M1+P1*Q1)/(1+P1*(P1*a3/a1)^(1/(r-1))...
    +P2*((P2*a3)/a2)^(1/(r-1)));
m2=(M2+P2*Q2)/(1+P1*(P1*b3/b1)^(1/(r-1))...
    +P2*((P2*b3)/b2)^(1/(r-1)));

money_mat(:,t)=money_mat(:,t-1);
money_mat(i,t)=m1;money_mat(j,t)=m2;

fprintf('Instance=%d, t=%d \n',inst,t);
end;

%store_money_mat(:,:,inst)=money_mat;

[y,x]=hist(money_mat(:,end),x_axis);
store_pdf(inst,:)=y./(sum(y)*unit);

end;

toc;
```

```
figure;
plot(x_axis,mean(store_pdf(1:inst,:),1),x_axis,exp(-x_axis));
xlim([0 5]);
```

Function file

```
function F = focfn(P)

global M1 M2 Q1 Q2 a1 a2 a3 b1 b2 b3 r

%Problematic syntax
%Fcheck =[(M1+P(1)*Q1)/(P(1)+P(2)*((P(2)*a1)/(P(1)*a2))^(1/(r-1))+(a1/(P(1)*a3))^(1/(r-1
%    (M2+P(2)*Q2)/(P(1)+P(2)*((P(2)*b1)/(P(1)*b2))^(1/(r-1))+(b1/(P(1)*b3))^(1/(r-1))) -
%    (M1+P(1)*Q1)/(P(2)+P(1)*((P(1)*a2)/(P(2)*a1))^(1/(r-1))+(a2/(P(2)*a3))^(1/(r-1)))+
%    (M2+P(2)*Q2)/(P(2)+P(1)*((P(1)*b2)/(P(2)*b1))^(1/(r-1))+(b2/(P(2)*b3))^(1/(r-1))) -

d11=P(1)+P(2)*((P(2)*a1)/(P(1)*a2))^(1/(r-1))+(a1/(P(1)*a3))^(1/(r-1));
d12=P(1)+P(2)*((P(2)*b1)/(P(1)*b2))^(1/(r-1))+(b1/(P(1)*b3))^(1/(r-1));
d21=P(2)+P(1)*((P(1)*a2)/(P(2)*a1))^(1/(r-1))+(a2/(P(2)*a3))^(1/(r-1));
d22=P(2)+P(1)*((P(1)*b2)/(P(2)*b1))^(1/(r-1))+(b2/(P(2)*b3))^(1/(r-1));



F = [( M1 + P(1) * Q1 ) + ( M2 + P(2) * Q1 ) - Q1 * d12 ;
      ( M1 + P(1) * Q1 ) + ( M2 + P(2) * Q2 ) - Q2 * d22 ] ;


end
```

5.2 Python codes

This set of codes are written by Sanjay Moorjani. The results are similar to the Matlab codes. However, occasionally differences appear in the results because of usage of different numerical solvers. As noted earlier, convergence is a big issue for solving the nonlinear market-clearing equations.

5.2.1 Binary trade

```
from scipy.optimize import fsolve
from sympy import symbols, diff
import matplotlib.pyplot as plt
import numpy as np
from scipy.interpolate import UnivariateSpline
from scipy.stats.kde import gaussian_kde
from numpy import linspace


M1=1
M2=1
Q1=1
Q2=1
a3=0.9
b3=a3
r=0.01

lowerBound=0
upperBound =0
binwidth = 0.05


numberOfAgents = 100
numberOfInstances = 1000

MoneyArray = np.ones(numberOfAgents)

max_iter = 1000

def FOC1(P1, P2, M1, M2):
return ((M1+M2+P1*Q1+P2*Q2) - (P1 + P2*((P2*a1)/(P1*a2))**(1/(r-1)) +  a3*((a1)

def FOC2(P1, P2, M1, M2):
return ((M1+M2+P1*Q1+P2*Q2) - (P2 + P1*((P1*a2)/(P2*a1))**(1/(r-1)) +  a3*((a2)
```

```
def equations(p):
    P1, P2 = p
    return (FOC1(P1,P2,M1,M2), FOC2(P1,P2,M1,M2))

moneyEvolution1 = []
moneyEvolution2 = []
total = []
store_moneyHistograms = []
# store_pdf=zeros(5/unit+1,num_av);
AverageMoneyArray =  np.zeros((1,100))
print(AverageMoneyArray)


for y in range(numberOfInstances):

MoneyArray = np.ones(numberOfAgents)
print(MoneyArray)

for x in range(max_iter):

a1= np.random.uniform(0,1-a3,1)[0]
b1= a1

a2=1-a1-a3
b2=1-b1-b3

agents= np.random.random_integers(numberOfAgents, size=(1,2))
# print(agents)

M1= MoneyArray[agents[0,0]-1]
# print(M1)
M2= MoneyArray[agents[0,1]-1]

P1, P2 =  fsolve(equations,(1,1))

# m1 = (1-a1-a2)*(M1 + P1*Q1)
# m2 = (1-b1-b2)*(M2 + P2*Q2)

m1 = a3*(M1 + P1*Q1)
m2 = b3*(M2 + P2*Q2)

MoneyArray[agents[0,0]-1]=m1
MoneyArray[agents[0,1]-1]=m2
```

```
if(min(MoneyArray)<lowerBound):
lowerBound=min(MoneyArray)

if(upperBound<max(MoneyArray)):
upperBound=max(MoneyArray)

data = MoneyArray
binwidth = 0.05
hist, binedges=np.histogram(data, bins=np.arange(0, 5 + binwidth, binwidth))
normalizedhist = hist/(hist.sum())

print(normalizedhist)
print(binedges)

store_moneyHistograms.append(normalizedhist)

finalHist=np.average(store_moneyHistograms, axis=0)
xAxis=np.arange(0, 5 , binwidth)
print(xAxis)
print(finalHist)

plt.bar(xAxis, finalHist, width=0.1)
plt.title("a3=0.9")
```

5.2.2 Binary trade within a population

```
from scipy.optimize import fsolve
from sympy import symbols, diff
import matplotlib.pyplot as plt
import numpy as np
from scipy.interpolate import UnivariateSpline
from scipy.stats.kde import gaussian_kde
from numpy import linspace


M1=1
M2=1
Q1=1
Q2=1
a3=0.1
b3=a3
r=10
```

```
lowerBound=0
upperBound =0
binwidth = 0.05


numberOfAgents = 100
numberOfInstances = 1

MoneyArray = np.ones(numberOfAgents)

max_iter = 1000

def FOC1(P1, P2, M1, M2):
return ((M1+M2+P1*Q1+P2*Q2) - (P1 + P2*((P2*a1)/(P1*a2))**(1/(r-1)) +  ((a1)/(P1*a3))**(

def FOC2(P1, P2, M1, M2):
return ((M1+M2+P1*Q1+P2*Q2) - (P2 + P1*((P1*a2)/(P2*a1))**(1/(r-1)) +  ((a2)/(P2*a3))**(

def equations(p):
    P1, P2 = p
    return (FOC1(P1,P2,M1,M2), FOC2(P1,P2,M1,M2))

moneyEvolution1 = []
moneyEvolution2 = []
total = []
store_moneyHistograms = []
# store_pdf=zeros(5/unit+1,num_av);
AverageMoneyArray =  np.zeros((1,100))
print(AverageMoneyArray)


for y in range(numberOfInstances):

MoneyArray = np.ones(numberOfAgents)
print(MoneyArray)

for x in range(max_iter):

a1= np.random.uniform(0,1-a3,1)[0]
b1= a1

a2=1-a1-a3
b2=1-b1-b3
```

```
agents= np.random.random_integers(numberOfAgents, size=(1,2))
# print(agents)

M1= MoneyArray[agents[0,0]-1]
# print(M1)
M2= MoneyArray[agents[0,1]-1]

print(M1,M2)

P1, P2 =  fsolve(equations,(1,1))

# m1 = (1-a1-a2)*(M1 + P1*Q1)
# m2 = (1-b1-b2)*(M2 + P2*Q2)

m1 = (M1 + P1*Q1)/( P1*((P1*a3/a1)**(1/(r-1))) + P2*((P2*a3/a2)**(1/(r-1))) + 1
m2 = (M2 + P2*Q2)/( P1*((P1*a3/a1)**(1/(r-1))) + P2*((P2*a3/a2)**(1/(r-1))) + 1

print(a1, a2, a3, P1, P2, m1, m2)


MoneyArray[agents[0,0]-1]=m1
MoneyArray[agents[0,1]-1]=m2

print(agents[0,0]-1,agents[0,1]-1)
print(MoneyArray.sum())
print(MoneyArray)

if(min(MoneyArray)<lowerBound):
lowerBound=min(MoneyArray)

if(upperBound<max(MoneyArray)):
upperBound=max(MoneyArray)

data = MoneyArray
binwidth = 0.05
hist, binedges=np.histogram(data, bins=np.arange(0, 5 + binwidth, binwidth))
normalizedhist = hist/(hist.sum()*binwidth)

print(data)
print(data.sum())
print(binedges)
print(normalizedhist)
store_moneyHistograms.append(normalizedhist)
```

```
finalHist=np.average(store_moneyHistograms, axis=0)
xAxis=np.arange(0, 5 , binwidth)
print(finalHist)



plt.bar(xAxis, finalHist, width=0.1)
plt.title("a3=0.1 r=10")
plt.plot(xAxis, np.exp(-xAxis),'r')
plt.show()
```

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