EQUILIBRIUM PRICING OF SPECIAL BEARER BONDS

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Abstract

Special Bearer Bonds provide immunity to investors in respect of black money invested in them. This paper derives equilibrium prices of these bonds in a continuous time framework using the mixed Wiener-Poisson process. The Capital Asset Pricing Model (CAPM) is modified to take into account the risk of tax raids for black money investors.

The pricing of all other assets relative to each other is shown to be unaffected by the presence of black money. This result extends the CAPM to capital markets like India where black money is widespread. Other applications include estimating the magnitude of black money.

Equilibrium Pricing of Special Bearer Bonds

1. Introduction

Special Bearer Bonds were made available for sale from the 2d February 1981 (vide Notification No. F 4(1)-W & M/81 dt. 15/1/81, 128 ITR 114). There were no application forms to be filled up for buying the bonds which are repayable to bearer. The bonds of a face value of Rs.10000 are redeemable after 10 years for Rs.12000. The premium on redemption is exempt from income tax, and the bonds themselves are exempt from wealth tax and gift tax. The most important provision is, however, the immunity conferred by Section 3 of the Special Bearer Bonds (Immunities and Exemptions) Act 1981 (7 of 1981) :

3. Immunities (1) Notwithstanding anything contained in any other law for the time being in force,

(a) no person who has subscribed to or has otherwise acquired Special Bearer Bonds shall be required to disclose, for any purpose whatsoever, the nature and source of acquisition of such bonds;

(b) no inquiry or investigation shall be commenced against any person under any such law on the ground that such person has subscribed to or has otherwise acquired Special Bearer bonds; and

(c) the fact that a person has subscribed to or has otherwise acquired Special Bearer Bonds shall not be taken into account and shall be inadmissible as evidence in any proceeding relating to any offence or the imposition of any penalty under any such law. S. 3(2) provides that the above immunity shall not extend to proceedings relating to theft, robbery, misappropriation of property, criminal breach of trust, cheating, corruption and similar offences; the immunity does not also cover civil liabilities (other than tax liabilities).

These provisions make the Special Bearer Bond (black bond for short) an attractive investment to those who fear tax raids or prosecutions. An active secondary market has also developed as the bonds, being payable to bearer, are transferable by delivery, thereby offering complete anonymity to buyer and seller alike.

The question that arises is how does this instrument get integrated into the capital market, and how is its price determined in this market.

2. The Model

To simplify the algebraic manipulation, we shall work in a continuous time framework. We assume that for the white investor (i.e., an investor who has no black money) the asset returns are generated by a Wiener process :

$$\frac{dP_j}{P_j} = \mu_j\, dt + dz_j \qquad (1)$$

The z_j are Wiener processes with drift zero and instantaneous covariances $\Sigma = [\sigma_{ij}]$. Thus the white investor sees an instantaneous return vector μ and instantaneous variance matrix Σ. We assume that there is a risk free asset (a white bond) which gives a return of R_{FW}.

The black investor sees things slightly differently: as long as there is no raid or investigation, the returns evolve according to a Wiener process as above; but if there is a raid leading to a detection of black money, taxes and penalties would be imposed leading to a negative return. We assume that the raids follow a Poisson process with parameter γ, and that conditional on a raid having taken place, the loss suffered by the investor is a fraction h of his wealth (excluding black bonds). The fraction h, which we shall call the grayness ratio, would vary from person to person depending on the fraction of wealth which is unaccounted for and the skill with which this black wealth has been concealed; it would also depend on the rates of tax and penalties leviable. We assume that the grayness ratio is always less than one; typical values would probably be in the range 0.1 to 0.5. (We can express h as $f_1f_2f_3$ where f_1 is the fraction of black wealth to total wealth, f_2 is the fraction of black wealth which will be detected during a raid and f_3 is the taxes and penalties as a fraction of the detected black wealth. Though f_3 could conceivably exceed one, the product $f_1f_2f_3$ must be less than one; else the individual will have to file for bankruptcy.) Under these assumptions, the returns accruing to the black investor would follow a mixed Wiener-Poisson process of the form:

$$\frac{dP_{jB}}{P_{jB}} = \mu_j\, dt + dz_j - h\, dq \qquad (2)$$

where q is a Poisson process with intensity γ. The Weiner and Poisson processes are independent of each other.

The instantaneous returns are now given by $\mu_j - h\gamma$; the instantaneous covariances by $\sigma_{ij} + h^2\gamma$. Letting e denote a vector of ones, the vector of instantaneous returns for the black investor can be written as $\mu - h\gamma e$; the instantaneous variance matrix is given by $\Sigma + h^2\gamma ee'$. The white bond is no longer risk free; its variance is $h^2\gamma$, and its covariance with other risky assets is also $h^2\gamma$; its mean return is $R_{FW} - h\gamma$. The risk free asset is the black bond which offers a return of R_{FB}. Under equilibrium, no white investor would hold a black bond, but black investors may hold the white bond.

We shall assume in all our analysis that investors have quadratic utility functions, or equivalently evaluate portfolio choices in a mean variance framework. Under this assumption, equilibrium returns on various assets must obey the well known Capital Asset Pricing Model (CAPM) developed by Tobin(1958), Sharpe(1964), Lintner(1965) and Mossin(196), and extended to continuous time by Merton(1973).

3. Equilibrium: A Simple Case

The simplifying assumption that we make in this section is that all investors are black; this means that the prices of all assets including the white bond are determined by the black investors. We also assume that all investors have the same grayness ratio h. This means that all investors see the same mean vector and covariance matrix of returns; of course, these are not the same as what a white investor would see, but there are no white investors. Under this condition, we have a traditional CAPM relationship between the means and betas as seen by the black investors. We will have :

$$E_B(R_j) - R_{FB} = E_B(R_{MB} - R_{FB})Cov_B(R_j, R_{MB}) / Var_B(R_M) = \lambda\, Cov_B(R_j, R_{MB}) \qquad (3)$$

where $\lambda = E_B(R_{MB} - R_{FB}) / Var_B(R_M)$

We use the subscript B with all the expectations, covariances and variances to emphasize that the stochastic processes to be used are those of the black investor; we write R_{MB} because the universe of risky assets for the black investor includes the white bond which is not part of the risky market portfolio as seen by a hypothetical white investor. We shall presently relate the quantities in Eqn. 3 to the corresponding quantities as seen by a hypothetical white investor.

Let c be the fraction of the black risky portfolio invested in assets other than the white bond (or equivalently, the fraction of white risky assets to all white assets); and let the subscript j denote any portfolio which does not contain black bonds. We then have :

$$R_{MB} = c\, R_M + (1-c)\, R_{FW} \qquad (4)$$

$$E_B(R_j) = E(R_j) - h\gamma \qquad (5)$$

$$Cov_B(R_j,R_{MB}) = c\, Cov(R_j,R_M) + h^2\gamma \qquad (6)$$

$$Var_B(R_{MB}) = c^2\, Var(R_M) + h^2\gamma \qquad (7)$$

$$Cov_B(R_j,R_{FW}) = h^2\gamma \qquad (8)$$

$$\lambda = [c\, R_M + (1-c)\, R_{FW} - h\gamma - R_{FB}] / [c^2\, Var(R_M) + h^2\gamma] \qquad (9)$$

$$R_{FW} = R_{FB} + h\gamma + \lambda\, h^2\, \gamma \qquad (10)$$

We can rewrite Eqns (9) and (10) as

$$\lambda = \frac{R_{FW} - h\gamma - R_{FB}}{h^2\gamma} = \frac{c[\ E(R_M) - R_{FW}] + R_{FW} - h\gamma - R_{FB}}{c^2 Var(R_M) + h^2\gamma} \qquad (11)$$

If x/y = (x+a)/(y+b) with b ≠ 0, then a/b = x/y. Hence, we have

$$\lambda = \frac{R_{FW} - h\gamma - R_{FB}}{h^2\gamma} = \frac{c[\ E(R_M) - R_{FW}]}{c^2 Var(R_M)} = \frac{E(R_M) - R_{FW}}{c\ Var(R_M)} \qquad (12)$$

and equation (10) becomes :

$$R_{FB} = R_{FW} - h\gamma - h^2\gamma\ [E(R_M) - R_{FW}] / c\ Var(R_M) \qquad (13)$$

Eqns. (13) expresses the dependence of the equilibrium black bond return on h and γ in terms of parameters familiar to our hypothetical white investor. To derive the pricing for other assets, we apply Eqn. (3) to the portfolio consisting of portfolio j fully financed by white borrowing (i.e. shorting the white bond). Using equations (5) and (6) and the fact that Cov(X-Y,Z) = Cov(X,Z) - Cov(Y,Z), we get :

$$E(R_j) - R_{FW} = c\ Cov(R_j, R_M)\ E_B(R_{MB} - R_{FB}) / Var_B(R_{MB}) \qquad (14)$$

In this equation, portfolio j can be the white risky market portfolio; substituting R_M for R_j and rearranging, we get :

$$E_B(R_{MB} - R_{FB}) = [E(R_M) - R_{FW}]\ Var_B(R_{MB}) / c\ Var(R_M) \qquad (15)$$

Substituting this value of $E_B(R_{MB} - R_{FB})$ into Eqn. (14) gives

$$E(R_j) - R_{FW} = \frac{Cov(R_j, R_M)}{Var(R_M)}\ E(R_M - R_{FW}) \qquad (16)$$

valid for any portfolio which does not contain black bonds. This is exactly the CAPM equation that a hypothetical white investor would write down if he were to completely ignore the existence of black bonds and black money, and compute all returns and betas in purely white terms using the white bond as the risk free asset. Here then is a market in which there is a black CAPM equation (Eqn. 3) which uses the black bond as the risk free asset and relates the returns as seen by the black investor to the betas as computed by him; this equation is valid for the black bond also. There is also a white CAPM equation (Eqn. 16) which uses the white bond as the risk free asset and relates the returns as seen by the white investor to the betas as seen by him ; this equation does not apply to the black bond.

The black and the white security analysts can certainly live in perfect harmony in this world without even being aware of each other's existence. But we still have to populate this world with white investors, and let the black investors assume various shades of gray; it is to this that we turn in the next section. Those readers to whom our derivation of the white CAPM equation looked like a piece of legerdemain can also have the pleasure (if such be it) of arriving at this result from first principles through the route of matrix algebra.

3. Equilibrium in the General Model

We now remove the assumption that all investors have the same grayness ratio. In particular, the grayness ratio of some investors could be zero; we allow white investors into the economy.

It is easily verified (see, for example, Merton(1973) that, under quadratic utility, the first order condition for utility maximization for investor k with wealth S^k, facing a mean vector μ^k, variance matrix Σ^k and risk free return r^k is :

$$\Sigma^k S^k d^k = g^k (\mu^k - r^k e) \qquad (17)$$

where g^k is the reciprocal of the investor's Arrow Pratt measure of risk aversion and d^k_j is the proportion of wealth invested in asset j (the investment in the risk free asset is $1-d^{k\prime}e$). The Arrow Pratt measure of risk aversion (Pratt, 1964) is equal to -U''(W)/U'(W) where U is the utility function for wealth.

In our case, $\mu - h^k\gamma e$ and $\Sigma + (h^k)^2\gamma ee'$ are the mean vector and variance matrix for the white risky assets as seen by investor k with grayness ratio h^k. In addition, the white risky asset must also be treated as a risky asset with mean return $R_{FW} - h\gamma$ and covariance $(h^k)^2\gamma$ with all risky assets. Thus Eqn. (17) takes the following form :

$$\left[\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} + (h^k)^2\gamma ee'\right] S^k \begin{bmatrix} p^k \\ b^k \end{bmatrix} = g^k \begin{bmatrix} \mu - h^k\gamma e - R_{FB}e \\ R_{FW} - h^k\gamma - R_{FB} \end{bmatrix} \qquad (18)$$

where we write d^k as $(p^{k\prime}, b^k)'$.

We can expand Eqn. (18) into two equations :

$$[\, \Sigma\, p^k + (h^k)^2\gamma ee'd^k\,]\, S^k = g^k\, [\mu - h^k\gamma e - R_{FB}e] \qquad (19)$$

$$(h^k)^2\gamma e'd^k\, S^k = g^k\, [R_{FW} - h^k\gamma - R_{FB}] \qquad (20)$$

To facilitate aggregation of the above equations over k we define :

$$h_2 = [\sum_k (h^k)^2 e'd^k S^k] / [\sum_k e'd^k S^k] \quad (21)$$

$$h = [\sum_k h^k g^k] / [\sum_k g^k] \quad (22)$$

$$d = [\sum_k d^k S^k] / [\sum_k S^k] \quad (23)$$

$$\delta = [\sum_k e'd^k S^k] / [\sum_k S^k] = e'd \quad (24)$$

$$p = [\sum_k p^k S^k] / [\sum_k S^k] \quad (25)$$

$$\pi = [\sum_k e'p^k S^k] / [\sum_k S^k] = e'p \quad (26)$$

$$g = \sum_k g^k \quad (27)$$

$$S = \sum_k S^k \quad (28)$$

Aggregation of Eqns. (19) and (20) gives :

$$\Sigma p + h_2\delta\gamma e = (g/S) [\mu - h\gamma e - R_{FB}e] \quad (29)$$

$$h_2\delta\gamma = (g/S) [R_{FW} - h\gamma - R_{FB}] \quad (30)$$

Substituting Eqn (30) into Eqn (29) gives

$$\Sigma p = (g/S) [\mu - R_{FW}e] \quad (31)$$

Since p/π is the weight of the white risky market portfolio, we can compute the covariances as seen by a white investor :

$$\mathrm{Cov}(R, R_M) = \Sigma p/\pi = [g/S\pi] [\mu - R_{FW}e] \quad (32)$$

giving the CAPM equation

$$\mu - R_{FW}e = \lambda \, \mathrm{Cov}(R, R_M) \quad (33)$$

or, in component (or portfolio) form,

$$E(R_j) - R_{FW} = \lambda \, \mathrm{Cov}(R_j, R_M) \quad (34)$$

where

$$\lambda = \pi S/g \quad (35)$$

Since Eqn (34) holds for the market portfolio also, we have:

$$\lambda = [E(R_M) - R_{FW}] / \mathrm{Var}(R_M) \quad (36)$$

so that the usual form of the CAPM equation obtains :

$$E(R_j) - R_{FW} = \beta_j [E(R_M) - R_{FW}] \tag{37}$$

Substituting Eqn.(35) into Eqn (30) we get

$$h_2 d\gamma = (\pi/\lambda) [R_{FW} - h\gamma - R_{FB}] \tag{38}$$

On using Eqn (36), this becomes

$$R_{FW} = R_{FB} + h\gamma + h_2\gamma [E(R_M) - R_{FW}] / c \text{ Var}(R_M) \tag{39}$$

where $c = \pi/\delta$ is the fraction of all white assets invested in risky white assets (i.e. assets other than the white bond).

This completes the analysis of the capital market in terms of white parameters.

We can also obtain a black version of the CAPM equation by aggregating Eqn (18) as follows :

$$\Sigma_B \, d = (g/S)(\mu_B - R_{FW}e) \tag{40}$$

where

$$\Sigma_B = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} + h_2\gamma ee' \qquad \mu_B = \begin{bmatrix} \mu - h\gamma e \\ R_{FW} - h\gamma \end{bmatrix} \tag{41}$$

Since h is a weighted average of h^k and h_2 is a weighted average of $(h^k)^2$, Σ_B and μ_B can be interpreted as the mean vector and variance matrix applicable to the average black investor (except that h_2 need not equal h^2). Now, d/δ is the weight of the market portfolio including white bonds; we can, therefore, derive a black CAPM equation as follows :

$$\text{Cov}_B(R, R_{MB}) = \Sigma_B d/\delta = [g/S\delta] [\mu_B - R_{FB}e] \tag{42}$$

$$\mu_B - R_{FB}e = \lambda_B \text{ Cov}_B(R, R_{MB}) \tag{43}$$

$$E(R_j) - R_{FB} = \lambda_B \text{ Cov}_B(R_j, R_{MB}) \tag{44}$$

where

$$\lambda_B = \delta S/g \tag{45}$$

Since Eqn (44) holds for the black market portfolio also, we have:

$$\lambda_B = [E(R_{MB}) - R_{FB}] / Var_B(R_{MB}) \quad (46)$$

so that the black CAPM equation holds :

$$E(R_j) - R_{FB} = \beta_{jB} [E(R_{MB}) - R_{FB}] \quad (47)$$

In equations (42) - (47), the vector R is extended to include the white bond also; further, j may be any portfolio whatsoever including black bonds.

We have thus derived the white CAPM equation (Eqn 37) and the black CAPM equation (Eqn 47) both of which explain the risk return relationship in the capital market. These correspond to and generalize Eqns (16) and (3) which we obtained in the simple model earlier. Once again the white CAPM equation does not apply to black bonds; however, Eqn. (39) which generalizes Eqn. (13) expresses the dependence of the equilibrium black bond return on h, h_2 and γ in terms of white parameters. The main differences between the results in this model and the earlier simpler model are:

(a) h is now a weighted average of the h^k, with the weights being the reciprocals of the Arrow-Pratt risk aversion coefficients;

(b) h_2 is a weighted average of the $(h^k)^2$ with the weights being the wealth invested in white assets; and

(c) h_2 need not equal h^2.

Of course, h and h_2 do not enter the white CAPM equation (Eqn. 37) at all.

Those readers who still finds it surprising that the white CAPM equation should hold in a market which has black investors (or even has only black investors) may find it useful to reflect on the following matrix identity :

$$\left[\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} + (h^k)^2 \gamma ee' \right]^{-1} = \begin{bmatrix} \Sigma^{-1} & (h^k)^2 \gamma \Sigma^{-1} e \\ (h^k)^2 (\Sigma^{-1} e)' & (1/(h^k)^2 \gamma) + e'\Sigma^{-1} e \end{bmatrix}$$

which follows from the formulas for partitioned inverses and for updating inverses after a rank one correction. The block of the inverse matrix corresponding to white risky assets continues to be Σ^{-1} regardless of the existence of black money risks. Moreover, the correction in the mean returns $h^k\gamma$ is the same for all white assets. These facts imply that black money should not affect the pricing of white risky assets inter se. Since the white risk free asset can be regarded as the limiting case of a white risky asset, the pricing of this relative to the other white risky assets should also be unaffected by the presence of black money. This indicates that the white CAPM equation should hold for all white assets. The only asset to which this argument does not apply is the black bond.

4. Conclusions and Implications

One important conclusion of this paper is that the presence of black money investors (who face the risk of tax penalties in addition to the normal investment risks) does not affect the pricing of white assets at all. This implies that the ordinary CAPM can continue to be used in all matters where black assets are not involved even if the white assets being considered are known to attract a lot of black investors. This provides justification for using the CAPM in corporate finance and portfolio management in a capital market like India where black money is widespread.

A simple relationship was shown to hold between the return on the black bond and that on the white bond :

$$R_{FW} = R_{FB} + h\gamma + h_2\gamma\, [E(R_M) - R_{FW}] \,/\, c\, Var(R_M) \qquad (36)$$

where c is the fraction of all white assets invested in risky white assets (i.e. assets other than the white bond), h and h_2 ($\div\, h^2$) represent the prevalence of black money in the economy, and γ represents the intensity of the Government's tax enforcement policies (frequency of raids).

Possible applications of this relationship include :

1. Black money investors could use this to decide on their policies relating to disposal of their black wealth. They could use the equation to estimate likely prices of the bond in future under alternative scenarios; they could also use current prices of the bonds to assess the market's perception of the parameter γ (the likelihood of tax raids) and use this as a crosscheck on their own judgment.

2. Researchers in economics and finance could use the prices of black bonds to estimate the parameter h (prevalence of black money) or parameter γ (intensity of tax enforcement) if the other parameter is known or can be independently estimated. More importantly, we can make an estimate of the change in the prevalence of black money (h) in any given period assuming that the tax enforcement parameter γ has not changed during this period (or using an independent estimate of the change in γ); alternatively, if an estimate of the change in the black money prevalence (h) is available, the change in the tax enforcement parameter can be estimated.

3. The Government could perhaps use this to arrive at a fair price at which any future issue of bearer bonds should be made. Typically, such issues are accompanied by an unannounced change in the tax enforcement parameter γ; the issue of bearer bonds itself (and the price at which it is issued) conveys information to the public about this change. This would complicate matters considerably.

APPENDIX

This appendix derives the means and variances as seen by a black investor in discrete time, and indicates how, as the time interval is reduced, the continuous time version is obtained.

If a random variable X is equal to a random variable X_1 with probability p and to another random variable X_2 with probability 1-p, then we have

$$E(X) = p\, E(X_1) + (1-p)E(X_2)$$

$$E\,[(X)^2] = p\, E\,[(X_1)^2] + (1-p)\, E\,[(X_2)^2]$$

$$[E(X)]^2 = p^2\, [E(X_1)]^2 + (1-p)^2\, [E(X_2)]^2 + p(1-p)E(X_1)E(X_2)$$

$$Var(X) = p\, Var(X_1) + (1-p)\, Var(X_1) + p(1-p)\, [E(X_1) - E(X_2)]^2$$

If $X_1 = (1-h)X_2$ then these simplify to

$$E(X) = (1-hp)E(X_2)$$

$$Var(X) = [p(1-h)^2 + (1-p)]\, Var(X_2) + p(1-p)h^2\, [E(X_2)]^2$$

$$= [1 - p\{1- (1-h)^2\}]\, Var(X_2) + p(1-p)h^2\, [E(X_2)]^2$$

If the random variable Y equals the random variable Y_1 when X equals X_1, and equals the random variable Y_2 when X equals X_2, then :

$$Cov(X,Y) = E(X,Y) - E(X)E(Y)$$

$$= p\, Cov(X_1,Y_1) + (1-p)\, Cov(X_2,Y_2) +$$

$$p(1-p)[E(X_1)(E(Y_1\text{-}EY_2) + E(X_2)(E(Y_2\text{-}EY_1)]$$

If $X_1 = (1-h)X_2$ and $Y_1 = (1-h)Y_2$ then this simplifies to

$$Cov(X,Y) = [1 - p\{1- (1-h)^2\}]\, Cov(X_2,Y_2) + p(1-p)h^2E(X_2)E(Y_2)$$

Consider a time interval t, and let the white investor's mean returns during this interval be $(1+r_jt)$ and the covariances be $\sigma_{ij}t$; let the probability of a raid during this time interval be γt. For the black investor, the mean returns are given by

$$(1+r_jt)(1-h\gamma t)$$

and the covariances are given by

$$\sigma_{ij}t[1 - \gamma t\{1- (1-h)^2\}] + \gamma t(1-\gamma t)h^2(1+r_it)(1+r_j)t).$$

If we substitute these values into Eqn. 3, we can obtain analogs of Eqns. (4) to (13); but the formulas are quite messy and difficult to use.

However, if t is small and we neglect terms of order t^2, the means become $[(1 + r_jt) - h\gamma t)$ and the covariances become $(\sigma_{ij}t + h^2\gamma t)$. In other words, the reduction of the means is roughly $h\gamma t$ and the increase in the covariances is roughly $h^2\gamma t$. This agrees with the continuous time formulation.

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